Farey Sequences Map Playable Nodes on a String*

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Abstract

Natural harmonics, *i.e.* partials and their harmonic series, may be isolated on a vibrating string by lightly touching specific points along its length. In addition to the two endpoints, stationary *nodes* for a given partial n present themselves at $n-1$ locations along the string, dividing it into n parts of equal length. It is not the case, however, that touching any one of these nodes will necessarily isolate the nth partial and its integer multiples. The subset of nodes that will activate the nth partial (termed *playable nodes* by the authors) may be derived by following a mathematically predictable pattern described by so-called *Farey sequences*. The authors derive properties of these sequences and connect them to physical phenomena. This article describes various musical applications: locating single natural harmonics, forming melodies of neighbouring harmonics, sounding multiphonic aggregates, as well as predicting the relative tuneability of just intervals.

1 Playable nodes isolate partials of a vibrating string

Consider a string fixed at both ends that is relatively thin with respect to its length L. Bowing or plucking disturbs the string by introducing a transverse wave, which causes the string to vibrate.[1] Since the endpoints of the string are fixed, they remain at rest. A point of rest along a vibrating string is called a *node* while the point of maximum displacement exactly halfway between two nodes is called an *anti-node*.

The wave moves along the string in both directions and reflects back inverted at both ends.[2] Reflected waves that are in phase with waves travelling in the opposite direction establish a waveform that is constant in time. This stable superposition of waves travelling

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[1] The frequency of a transversely vibrating string depends on its length L and the speed of propagation of the wave v, related to the tension T and mass per unit length μ as described by Mersenne's law: $f = \frac{v}{2L} \equiv \frac{1}{2L}\sqrt{\frac{T}{\mu}}$. Note it is also possible to induce longitudinal vibration in a string by stroking (*i.e.* bowing parallel to the string), a technique Ellen Fullman uses to play her *Long String Instrument*. In the case of longitudinal vibration, propagation depends rather on small differences in local tension (stiffness) related to the string's modulus E and not the mean tension T; thus, in this case the frequency depends merely on the string's length and its physical properties: $f = \frac{v}{2L} \equiv \frac{1}{2L}\sqrt{\frac{E}{\mu}}$.

[2] The amplitude of the vibration diminishes over time due to friction unless energy is continuously reintroduced into the system, *e.g.* by means of bowing.

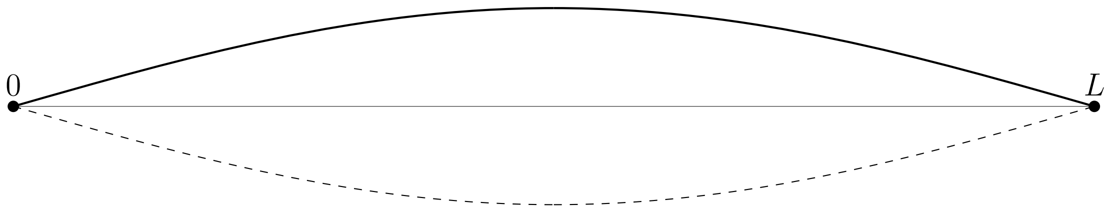

Figure 1: The ends on the string form nodes while the point of maximum displacement in the middle of the string forms an anti-node.

in opposite directions is called a standing wave because the string appears to be 'standing still'. If the only nodes present in the standing wave are the endpoints, then the string is vibrating in its fundamental mode, producing the 1st partial[3] with frequency f_1 (Figure 1). This frequency is referred to as the *fundamental frequency* and is often heard as the pitch of the string.[4] In the fundamental mode, a single propagation along the string only produces a half period of the sinusoidal waveform, so the wavelength of the 1st partial is twice the length of the string ($2L$).

Any division of the string into a whole number of parts produces a possible mode of vibration. For example, dividing the string into b equal parts places nodes at distances $\frac{L}{b}$ apart $(0, \frac{1}{b}L, \frac{2}{b}L, ..., \frac{b}{b}L = L)$. These nodes take the form $\frac{a}{b}L$ and the mode of vibration produces the bth partial with frequency $f_b = bf_1$. Caspar Johannes Walter has summarised the function of these fractions in practical terms, writing '[t]he denominator defines ... the [resulting] pitch and the numerator makes the position of the node along the string concrete.'[5]

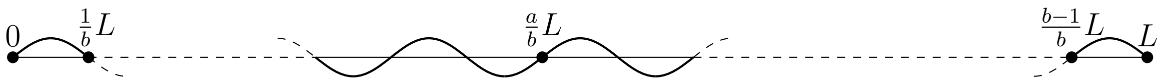

Figure 2: Mode of a the bth partial produced by lightly touching the string at $\frac{a}{b}L$.

Touching the string at $\frac{a}{b}L$ dampens all modes of vibration that do not have a node at $\frac{a}{b}$. Only modes that allow the string to be at rest at $\frac{a}{b}$ may form standing waves and, in most cases, the perceived pitch of the string will change according to the lowest partial possible under these conditions. For example, touching at $\frac{1}{2}L$ dampens the 1st and all other odd partials, each of which has an anti-node at $\frac{1}{2}L$, while allowing the 2nd partial and its multiples to vibrate. The pitch of the string sounds an octave higher because the

[3]In this article, when the authors refer to producing or sounding "partials" of a string, the term is understood as referring to the fundamental frequency perceived. These sounds are actually aggregates consisting of a partial along with its integer multiples, forming an harmonic series.

[4]Partials are individual sinusoidal frequency components of a non-sinusoidal sound. Aggregates that consist of whole number multiples of a single, generating fundamental frequency (present or not) are perceived as *harmonic sounds* because they form part of one harmonic series. Partials of a vibrating string are, for the most part, harmonic.

[5]Caspar Johannes Walter, 'Mehrklänge auf dem Klavier. Vom Phänomen zur Theorie und Praxis mikrotonalen Komponierens', in Pätzold and Walther (eds.), *Mikrotonalität – Praxis und Utopie*, SMS Band 3 (Mainz: Schott Music, 2014), p. 16. From original: 'Der Nenner definiert also die Tonhöhe und der Zähler konkretisiert die Position des Schwingungsknotens auf der Saite..'

wavelength has been halved, causing a doubling of f_1.[6] Likewise, by touching at $\frac{1}{3}L$ or $\frac{2}{3}L$, both the 1st and 2nd partials are dampened while the 3rd partial and its multiples may sound. The sounding pitch is a perfect twelfth higher with frequency $3f_1$.

Only integer ratios $\frac{a}{b}$ that are in lowest terms, *i.e.* where a and b are coprime, will isolate partial b and its harmonics; these are called *playable nodes.*[7] If $\frac{a}{b}$ were not in lowest terms, then a simpler mode of vibration at that node would be possible, producing partial $\frac{b}{\gcd(a,b)}$ with frequency $\frac{b}{\gcd(a,b)}\, f_1$. For example, the 4th partial will only sound at $\frac{1}{4}L$ and $\frac{3}{4}L$ but not at $\frac{2}{4}L \equiv \frac{1}{2}L \;\Rightarrow$ 2nd partial.

If the string is divided into b parts and b is a prime, then lightly touching each of the $b-1$ nodes (*i.e.* excluding endpoints) will sound partial b. For any other (nonprime) values of b, all nodes that coincide with simpler modes of vibration must be subtracted. The exact number of playable nodes that produce partial b is calculated by Euler's totient function (b), which counts the number of positive integers a less than b that are relatively prime with b (or, equivalently, the number of positive integers a less than b such that $\frac{a}{b}$ is in lowest terms).

As successively higher partials are activated, playable nodes fall increasingly close together when compared to the width of a finger producing harmonics and to the thickness of the string in question. Gradually, it becomes more and more difficult to play clear partials, especially on shorter, thicker strings. Instead, aggregates – clusters of neighbouring partials called *multiphonics* – begin to emerge. What form do these take?

In 'Mehrklänge auf dem Klavier. Vom Phänomen zur Theorie und Praxis mikrotonalen Komponierens',[8] Walter sketches a possible mathematical model of partials and multiphonics on strings.[9] He bases his theory on empirical observations from which he has inferred a procedure that invokes the *Fibonacci sequence* and a further process of transformative branching that he calls *mutation*.

Walter correctly recognises a pattern that emerges when ordering playable nodes. Beginning with the boundaries of a string at $\frac{0}{1}$ and $\frac{1}{1}$, progressive divisions may be listed systematically by calculating new divisions, called mediants, which would lie between each pair of already existing fractions. The mediant is equal to the sum of the two numerators divided by the sum of the two denominators, sometimes called the 'freshman's sum' of the two fractions. Because new terms of the Fibonacci sequence are also generated by summing the previous two terms, Walter associates this 'addition' of fractions with an expansion of Fibonacci terms in the numerators and denominators. This method produces sequences of fractions in which each new term lies between the preceding two, a back-and-forth 'zig-zag' of nested intervals narrowing around a particular location where a multiphonic may possibly be produced.

[6]Mersenne's law may be reformulated to relate frequency to the speed of propagation of the wave along the string and wavelength: $f = \frac{v}{\lambda}$. Since v is constant for a given string (depending on its tension and density), frequency is affected only by wavelength.

[7]Here the term *playable* is used to mean *potentially playable* on an ideal string. Clearly, higher partials become increasingly difficult to play on a real string; in practice, the limit of playability depends on physical characteristics of the string, *e.g.* its length and material properties.

[8]'Multiphonics on the Piano. From Phenomenon to Theory and Practice of Microtonal Composing'

[9]Walter, 'Mehrklänge auf dem Klavier', pp. 13–40.

For example, the sequence $\left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}\right\}$ follows the 'Fibonacci rule' to calculate numerators and denominators and leads to a particular multiphonic combination. However, to complete the set of all possible multiphonic combinations of playable nodes up to and including 8 requires three additional sequences: those leading to $\frac{1}{8}$, $\frac{3}{8}$, and $\frac{7}{8}$ respectively. Generating each of these sequences follows a different mutation in Walter's framework, where mediants are generated from two previous, but *non-successive* terms in the Fibonacci sequence.

$$\left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}\right\}$$

$$\left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}\right\}$$

$$\left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}\right\}$$

The first of these sequences mutates after $\frac{1}{2}$, at which point each new term is generated by calculating the freshman's sum of the previous term with the first term in the sequence, $\frac{0}{1}$. The second sequence initially follows this same mutation until $\frac{1}{3}$, at which point the usual 'summing' of the preceding two terms is resumed. The third sequence mutates after $\frac{1}{2}$, at which point new terms are generated by finding the freshman's sum of the preceding term with $\frac{1}{1}$.

Reformulating Walter's observations through properties of *Farey sequences*, which systematically organise the fractions and their mediants in increasing order, makes it possible to more easily develop the theory of playable nodes in a rigorous and transparent way. This article discusses a theoretical mapping of string divisions based on these structures and derives various musical applications.

2 Properties of Farey sequences

A Farey sequence F_k with of degree k is the set of reduced fractions $\frac{a}{b}$ between 0 and 1 whose denominators b do not exceed k, listed in strictly increasing order. The first four Farey sequences may be expressed in the following form.

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$

$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$

$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

$$F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$

Farey sequences have some unique properties:

a) any pair of consecutive fractions $\left(\frac{a}{b}, \frac{p}{q}\right)$ in a Farey sequence is called a 'Farey pair'; all Farey pairs have the property that $pb - aq = 1$, implying that the denominators b and q as well as the numerators a and p are relatively prime, *i.e.* share no common factors;

b) any Farey pair $\left(\frac{a}{b}, \frac{p}{q}\right)$ first appears in $F_{\max(b,q)}$; these two fractions remain consecutive in all subsequent F_ks until separated by their mediant $\frac{m}{n} = \frac{a+p}{b+q}$ in the Farey sequence F_{b+q};

c) once a Farey pair is separated by its mediant, the mediant forms a new Farey pair with each of the 'parent' fractions;

d) the mediant of a Farey pair is in lowest terms;

e) the mediant of a Farey pair $\left(\frac{a}{b}, \frac{p}{q}\right)$ is the fraction with the smallest denominator lying between $\frac{a}{b}$ and $\frac{p}{q}$;

f) Farey sequences divide the interval between 0 and 1 symmetrically, *i.e.* every fraction $\frac{a}{b}$ has a complement $1 - \frac{a}{b}$ appearing in the same sequence.

Proofs for these properties are presented in Appendix A.

3 Melodies – intervals – chords

3.1 The Farey sequence F_k maps the playable nodes for all partials $b \leq k$ of a vibrating string

Since a Farey sequence F_k lists all reduced fractions $\frac{a}{b}$ between 0 and 1 with denominators k or smaller in ascending numerical order, F_k also lists all playable nodes along a string that produce partials up to and including k. The playable nodes for a given partial k are first enumerated in the Farey sequence F_k. Properties of Farey sequences may therefore be applied to playable nodes:

- any two neighbouring[10] playable nodes in F_k produce partials that are coprime; conversely, given two coprime partials, it is possible to determine where along the string they have neighbouring playable nodes (see Subsection 3.2 below);
- between any two neighbouring playable nodes with coprime partials b and q, the lowest partial with a playable node between them is $b + q$;
- every pair of neighbouring playable nodes appears symmetrically from either end of the string.

[10] *I.e.* neighbouring in some Farey sequence F_k; as k increases, the physical distance between neighbouring playable nodes reduces.

Playable nodes may be interpreted musically in a number of ways. For example, while lightly gliding along a vibrating string, it is possible to produce melodies traversing neighbouring partials.

Consider the highest string of a contrabass (G2) with vibrating length 1050 mm from nut to bridge. A player may slide from the 2nd partial produced at $\frac{1}{2}L$ in either direction, reaching the 3rd partial at $\frac{1}{3}L$ or $\frac{2}{3}L$ respectively.[11] The lowest partial occurring between the 2nd and 3rd is their sum, partial 5, though many other higher partials may also be sounded along the way. Choosing some arbitrary limit, *e.g.* 19, the Farey sequence F_{19} gives an ordered list of all playable nodes up to the 19th partial. The denominators between $\frac{1}{3}$ and $\frac{1}{2}$ or $\frac{1}{2}$ and $\frac{2}{3}$ describe a melody of partials that may be sounded when sliding between the 2nd and 3rd partials. By symmetry, the melody from $\frac{1}{2}$ to $\frac{2}{3}$ is the retrograde of the melody from $\frac{1}{3}$ to $\frac{1}{2}$; each melodic interval is expressed by the ratio between successive denominators.

$$\left\{\frac{1}{3}, \frac{6}{17}, \frac{5}{14}, \frac{4}{11}, \frac{7}{19}, \frac{3}{8}, \frac{5}{13}, \frac{7}{18}, \frac{2}{5}, \frac{7}{17}, \frac{5}{12}, \frac{8}{19}, \frac{3}{7}, \frac{7}{16}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}, \frac{7}{15}, \frac{8}{17}, \frac{9}{19}, \frac{1}{2}\right\}$$

$$\left\{\frac{1}{2}, \frac{10}{19}, \frac{9}{17}, \frac{8}{15}, \frac{7}{13}, \frac{6}{11}, \frac{5}{9}, \frac{9}{16}, \frac{4}{7}, \frac{11}{19}, \frac{7}{12}, \frac{10}{17}, \frac{3}{5}, \frac{11}{18}, \frac{8}{13}, \frac{5}{8}, \frac{12}{19}, \frac{7}{11}, \frac{9}{14}, \frac{11}{17}, \frac{2}{3}\right\}$$

Multiplying each of these fractions by the vibrating length ($L = 1050$ mm) gives the positions of these playable nodes on a contrabass (see Appendix B for a notated example with string lengths). Since each Farey pair $\left(\frac{a}{b}, \frac{p}{q}\right)$ satisfies the property $pb - aq = 1$, the distance between them is $\frac{p}{q} - \frac{a}{b} = \frac{pb-aq}{qb} = \frac{1}{qb}$. Thus, the physical distance on a string between playable nodes with denominators q and b is $\frac{1}{qb}L$. In other words, the distance between two neighbouring (relatively prime) partials is the string length divided by their product. In the example above, partials 3 and 17, producing the melody 3:17, are separated by $\frac{1050 \text{ mm}}{3\times17} = \frac{1050 \text{ mm}}{51} = 21$ mm (see Appendix B).

3.2 The extended Euclidean algorithm locates neighbouring nodal positions of any two coprime partials

The definition of a Farey pair $pb - aq = 1$ has the shape of Bézout's identity for coprime integers b and q, which expresses the greatest common divisor of b and q as a linear combination of b and q multiplied by *integer* coefficients x_1 and x_2.

$$x_1 b + x_2 q = \gcd(b, q) = 1$$

The extended Euclidean algorithm may be used to calculate the coefficients of Bézout's identity. The definition of a Farey pair 'converts' the coefficients into the numerators of a Farey pair with denominators b and q, first appearing in the Farey sequence $F_{\max(b,q)}$.

[11] Since Farey sequences are symmetric, it may be assumed, for simplicity, that $\frac{1}{3}L$ and $\frac{2}{3}L$ refer to fractions of the string measured from the bridge. Thus, when a node is stopped, the fraction will refer to the sounding length.

This indicates where along a string coprime partials b and q have neighbouring playable nodes.[12]

As an illustration, consider partials 5 and 13. The extended Euclidean algorithm is employed to calculate Bézout's identity.

STEP A. Express the larger number (13) as a multiple of the smaller number (5) plus a remainder R_A.

$$13 = (2)5 + 3$$

STEP B. Repeat the process by expressing the smaller number (5) as a multiple of R_A plus a new remainder R_B.

$$5 = (1)3 + 2$$

STEP C. Repeat the process by expressing R_A as a multiple of R_B plus a new remainder R_C.

$$3 = (1)2 + 1$$

Once the remainder is equal to 1, which eventually happens whenever the two initial values are coprime, the algorithm stops.

Each step above may be rearranged to express the remainder as a sum of the other terms. The coefficient '1' is added where needed such that each term has a coefficient.

$$\begin{aligned} &\textbf{STEP A.} \quad (1)13 + (-2)5 = 3 \\ &\textbf{STEP B.} \quad (1)5 + (-1)3 = 2 \\ &\textbf{STEP C.} \quad (1)3 + (-1)2 = 1 \end{aligned}$$

This rearranged form allows for the construction of Bézout's identity. Working backward, earlier lines of the algorithm may be successively substituted to express the final remainder ($R_C = 1$) in terms of a difference of the original integers (5 and 13) multiplied by coefficients.

In **STEP C**, 2 may be replaced by the expression in **STEP B**.

$$(1)3 + (-1)[\,5 + (-1)3\,] = 1$$
$$\Rightarrow (2)3 + (-1)5 = 1$$

Similarly, 3 may be replaced by the expression in **STEP A**, reducing **STEP C** to a linear combination of 5 and 13.

$$(2)[\,(1)13 + (-2)5\,] + (-1)5 = 1$$
$$\Rightarrow (2)13 + (-5)5 = 1$$
$$\Rightarrow (2)13 - (5)5 = 1$$

The definition of a Farey pair is $pb - aq = 1$. In this example, denominators b and q are assigned to partials 13 and 5 respectively and numerators p and a are assigned to the

[12]That is, there are no playable nodes isolating partials less than min(b,q) lying between them.

coefficients 2 and 5. For a string with length L, partials 5 and 13 will have neighbouring nodes at $\frac{5}{13}L$ and $\frac{2}{5}L$, forming the Farey pair $\left(\frac{5}{13}, \frac{2}{5}\right)$ in all Farey sequences F_k where $13 \leq k < (5 + 13)$.[13] For a guitar with vibrating length *ca.* 635 mm, the distance between these neighbouring playable nodes is approximately 10 mm, while the distance on an average contrabass with vibrating length *ca.* 1050 mm is *ca.* 16 mm.

3.3 Branches of mediants in Farey sequences describe the components of multiphonics

A *multiphonic* is an activation of several neighbouring playable nodes. It is facilitated by making a broader point of contact with the string in the region around these nodes and by plucking or bowing the string at a point maximising energy of the desired partials.[14] The balance of components in the sound complex – partials, noise – is affected by both *where* and *how* the string is activated (finger position, plucking or bowing position, pressure, velocity). Production of multiphonics is also influenced by physical resistances within the string caused by its material qualities, *e.g.* stiffness, thickness, *etc.* These may also cause a certain degree of distortion in the harmonic spectrum of the string, slightly raising ('stretching') the actual frequencies of (some) higher partials.

Despite these variations, the Farey sequence ordering holds and, therefore, may be used to predict the structure of possible multiphonics. In the simplest case, these take the form $a : (a + b) : b$ where a and b are coprime partials. Erring on either side of the mediant $(a + b)$ will favour either $(2a + b)$ or $(a + 2b)$ respectively, suggesting two possible four-pitch aggregates for any coprime a and b.

$$a : (2a + b) : (a + b) : b$$

$$a : (a + b) : (a + 2b) : b$$

It would be useful in further studies to determine how various real-world parameters affect these predictions. Which positions on what kinds of strings favour the production of multiphonics? What is the relationship between overall string length, the size of the point of contact (fingertip or other object), and highest partials isolated? How do the frequency components of multiphonics produced at similar positions on a variety of strings and instruments vary?

3.4 Farey sequences define an harmonic space of stopped pitches

By Mersenne's law, a string of length L vibrates with wavelength $2L$ and frequency $f_1 = \frac{v}{2L}$. If a playable node at $\frac{a}{b}$ is stopped, the length of string vibrating is $\frac{a}{b}L$ with wavelength $\frac{a}{b} \times 2L$ and with frequency $v \div (\frac{a}{b} \times 2L) = \frac{b}{a} \times \frac{v}{2L}$. Thus, stopping the string at a playable node will sound a pitch above the open string with frequency ratio equal to

[13] Additionally, because Farey sequences are symmetric, there is a complementary Farey pair that also satisfies this condition at $\left(1 - \frac{2}{5}, 1 - \frac{5}{13}\right) \Rightarrow \left(\frac{3}{5}, \frac{8}{13}\right)$. Note that $(\frac{a}{b}, \frac{p}{q})$ and its complement $(1 - \frac{p}{q}, 1 - \frac{a}{b})$ are clearly equivalent by symmetry, depending merely on the end of the string from which one measures.

[14] The ideal point of contact imparting energy to a given partial b is around the antinode, *i.e.* at a distance between $\frac{1}{3}(\frac{1}{b})$ and $\frac{2}{3}(\frac{1}{b})$ from the bridge.

the inverse of the Farey fraction, namely $\frac{b}{a}$. The frequency of this pitch will be $\frac{b}{a}\,f_1$. For example, stopping at $\frac{5}{12}L$ (measured from the bridge) will produce a pitch with frequency twelve-fifths times that of the open string. If the contrabass' G string is tuned to 98 Hz (when A4 is tuned to 441 Hz), this pitch will be tuned to $\frac{12}{5} \times 98 = 235.2$ Hz.

Moving from one stopped playable node to the next produces a microtonal scale in which every musical interval is epimoric.[15] Since for Farey neighbours $pb - aq = 1$,

$$\frac{p}{q} \div \frac{a}{b} = \frac{pb}{aq} = \frac{aq+1}{aq}.$$

Notice that in Farey sequence F_k, the largest denominator is k and the largest numerator is $k-1$. But, if one fraction has denominator k, then its Farey neighbour cannot have numerator $k-1$ since its denominator would then have to be k as well and k is not coprime with itself. The largest possible a is therefore $k-2$ with $aq = k(k-2)$. The smallest melodic step between stopped playable nodes in F_k lies between $\frac{k-2}{k-1}$ and $\frac{k-1}{k}$, namely $\frac{k(k-2)+1}{k(k-2)}$.

In the early 2000s, inspired by the experience of playing James Tenney's *Koan*, Marc Sabat undertook an empirical investigation of microtonal intervals on string instruments, comparing the sound of all lowest terms ratios of numbers up to and including 28.[16] The intention was to establish a set of just intonation intervals tuneable directly by ear. Such intervals allow musicians playing non-fixed-pitch instruments to reliably establish microtonal pitches and provide a possible psychoacoustic basis for just intonation (JI) composition. Tuneable intervals are constrained by various factors affecting the perception of harmonic sound: *timbre* (harmonic, spectrally rich sonorities are easier to tune); *register* (the periodicity pitch or common fundamental should be generally above *ca.* 20 Hz and the common partial below *ca.* 4000 Hz); and *size* (interval width should be larger than a tone due to critical band effects and generally smaller than *ca.* 3 octaves due to segregation of the frequencies). Tuneability of intervals correlates, in part, with the timbral sound of 'spectral fusion'.

Farey sequences offer a systematic method of ordering candidate tuneable intervals at stopped playable nodes $\frac{b}{a}$ and, given any two neighbouring ratios, finding the 'next-simplest' JI ratio that lies between them: their mediant.

For example, consider the equal-tempered major third T with interval size equal to 400 cents.[17] T is nested between $\frac{2}{1}$ (octave, 1200 cents) and $\frac{1}{1}$ (open string or unison, 0 cents). By comparing T with the mediant of these two fractions, it is clear that T lies between $\frac{3}{2}$

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[15]The interval between two frequency ratios is determined by dividing them. An epimoric or superparticular ratio takes the form $\frac{n+1}{n}$. In musical terms, this is the interval between two successive partials in an harmonic series.

[16]Marc Sabat, *23-Limit Tuneable Intervals above and below A* (Berlin: Plainsound Music Edition, 2005).

[17]Mathematician Alexander J. Ellis (1814–1890) proposed the division of each equal tempered semitone into very small, equal units of measure. By dividing each semitone into 100 units called 'cents', the octave is pixelated into an equal division scale of 1200 parts, which may be used as a kind of ruler to measure and compare the absolute sizes of intervals. For a stopped pitch on a string at division $\frac{b}{a}$, its size in cents is defined as $1200 \times \log_2(\frac{b}{a})$.

(perfect fifth, 702 cents) and $\frac{1}{1}$. The sequence of nested bounds proceeds as follows.

$$\left(\frac{4}{3},\frac{1}{1}\right) \to \left(\frac{4}{3},\frac{5}{4}\right) \to \left(\frac{9}{7},\frac{5}{4}\right) \to \left(\frac{14}{11},\frac{5}{4}\right)$$

Since the interval $\frac{14}{11}$ is no longer directly tuneable[18] and all subsequent mediants will be non-tuneable ratios of even larger numbers, the nearest tuneable intervals to T are $\frac{5}{4}$ (386 cents) on the smaller side and $\frac{9}{7}$ (435 cents) on the larger side. The size of this *tolerance region* between neighbouring tuneable intervals is the microtonal interval $\frac{36}{35}$ (49 cents, approximately a quartertone). To find the 'simplest' frequency ratios nesting 400 cents within a practically imperceptible tolerance of 2 cents, the process may be continued.

$$\left(\frac{19}{15},\frac{5}{4}\right) \to \left(\frac{24}{19},\frac{5}{4}\right) \to \left(\frac{29}{23},\frac{5}{4}\right) \to \left(\frac{29}{23},\frac{34}{27}\right)$$

Comparing the size of these last two ratios, $\frac{29}{23}$ has 401 cents and $\frac{34}{27}$ has 399 cents and the size of the region they enclose is $\frac{783}{782}$ (2 cents), well below experimental thresholds for just noticeable difference, measured at approximately 3 Hz for tones around 500 Hz, equivalent to *ca.* 6–10 cents.[19]

Stopped playable nodes in the range between $\frac{1}{1}$ and $\frac{1}{2}$ may also be considered as a gamut of octave-equivalent, 'Monophonic' pitch-classes in the sense introduced by Harry Partch.[20] Since Farey sequences introduce numerically larger frequency ratios in a systematic way, it could be compositionally fruitful to consider successive F_ks filtered by prime factors (or up to a certain prime limit, like Partch's 11-limit scale) to generate subsets suggesting musically interesting proximities and enharmonic moves within harmonic space.

[18]Sabat, *23-Limit Tuneable Intervals.*

[19]B. Kollmeier, T. Brand, and B. Meyer, 'Perception of Speech and Sound', in Benesty, Sondhi, and Huang (eds.), *Springer handbook of speech processing*, (New York: Springer, 2008), p. 65.

[20]Harry Partch, *Genesis of a Music*, 2nd Edition (Boston: Da Capo Press, 1979).

Appendices

A Mathematical proofs: Farey Sequences

Though properties of Farey sequences have already been derived in a number of elegant and rigorous ways, the authors devised the following proofs to clarify specific results referred to in this text. We wish to acknowledge the excellent and comprehensive mathematical contributions to this domain in Hardy and Wright,[21] Graham, Knuth, and Pataschnik,[22] *et al.*

Given a positive integer k, consider the collection F_k of all rational numbers $\frac{a}{b}$ where $0 \leq \frac{a}{b} \leq 1$, $b \leq k$, and $\frac{a}{b}$ is in lowest terms. Only a finite number of fractions satisfies these conditions for each k and, being in lowest terms, they are unique. Therefore, F_k may be written in strictly increasing order and is called the Farey sequence of order k.

$$F_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\}$$

$$F_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\}$$

$$F_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\}$$

$$F_4 = \left\{ \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \right\}$$

For $k \leq 4$ in the preceding examples, it appears that consecutive fractions in Farey sequences, *i.e.* any neighbours $\frac{a}{b}$ and $\frac{p}{q}$ where $\frac{a}{b} < \frac{p}{q}$, satisfy the property $pb - aq = 1$.

Definition 1. Let any two fractions $\frac{a}{b} < \frac{p}{q}$ that satisfy the property $pb - aq = 1$ be called a Farey pair.

Definition 2. Define the mediant $\frac{m}{n}$ of $\frac{a}{b}$ and $\frac{p}{q}$ as

$$\frac{m}{n} = \frac{a+p}{b+q}.$$

Theorem. Any two neighbouring fractions in any Farey sequence F_k form a Farey pair. The proof will proceed by induction. Note that $\frac{0}{1}$ and $\frac{1}{1}$ satisfy the definition of a Farey pair in F_1, so the Theorem holds for F_1.

Lemma 1. Let $\frac{a}{b}$ and $\frac{p}{q}$ be a Farey pair. It follows that all of the following pairs of integers are coprime: (a, b), (p, q), (a, p), and (b, q). In particular, $\frac{a}{b}$ and $\frac{p}{q}$ are in lowest terms.

[21] G.H. Hardy and E.M. Wright, 'The Farey Series and a Theory of Minkowski', *An Introduction to the Theory of Numbers*, 4th Edition (London: Oxford University Press, 1975), pp. 23–37.

[22] R.L. Graham, D.E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd Edition (Boston: Addison-Wesley, 1989), pp. 115–139.

Proof. Let c be a positive integer that divides both a and b; then $p\left(\frac{b}{c}\right) - \left(\frac{a}{c}\right)q$ is an integer. Since $pb - aq = 1$, then $\frac{pb-aq}{c} = \frac{1}{c} \Rightarrow c = 1$. The same argument holds for each of the other pairs.

Lemma 2. Let $\frac{a}{b}$ and $\frac{p}{q}$ be a Farey pair, and let $\frac{m}{n}$ be their mediant.
The following properties hold true.

$$\left(\frac{a}{b}, \frac{m}{n}\right) \text{ and } \left(\frac{m}{n}, \frac{p}{q}\right) \text{ are Farey pairs} \tag{1}$$

$$\frac{a}{b} < \frac{m}{n} < \frac{p}{q} \tag{2}$$

$$\frac{m}{n} \text{ is in lowest terms} \tag{3}$$

$$\text{if some } \frac{r}{s} \text{ in lowest terms satisfies } \frac{a}{b} < \frac{r}{s} < \frac{p}{q}, \text{ then } s \geq n \tag{4}$$

An obvious consequence of Properties (1) and (4) is that there cannot be two lowest terms fractions with the same denominator nested inside a Farey pair.

Proof (1). Property (1) follows from the Farey pair definition because

$$mb - an = (a+p)b - a(b+q) = pb - aq = 1$$

and

$$pn - mq = p(b+q) - (a+p)q = pb - aq = 1.$$

Proof (2). Property (2) follows from property (1) because

$$\frac{m}{n} - \frac{a}{b} = \frac{mb - an}{nb} = \frac{1}{nb} > 0$$

and

$$\frac{p}{q} - \frac{m}{n} = \frac{pn - mq}{qn} = \frac{1}{qn} > 0.$$

Proof (3). Property (3) follows from Lemma 1 and property (1) because all fractions that form Farey pairs are in lowest terms.

Proof (4). Property (4) follows because

$$\frac{p}{q} - \frac{a}{b} = \frac{pb - aq}{qb} = \frac{1}{qb}$$

$$\Rightarrow \frac{1}{qb} = \left(\frac{p}{q} - \frac{r}{s}\right) + \left(\frac{r}{s} - \frac{a}{b}\right) = \frac{ps - rq}{qs} + \frac{rb - as}{sb}$$

.

Since both terms in the sum are positive by the definition of $\frac{r}{s}$ in Property (4), as are qs and sb, $ps - rq$ and $rb - as$ must also be positive integers, and thus

$$\frac{1}{qb} \geq \frac{1}{qs} + \frac{1}{sb} = \left(\frac{b+q}{s}\right)\left(\frac{1}{qb}\right)$$
$$\Rightarrow s \geq b + q = n.$$

Proof of the Theorem. Proceeding by induction, we assume for some $k \geq 1$ that all neighbouring fractions in F_k are Farey pairs. Let $n = k + 1$ and let

$$0 < \frac{m}{n} < 1$$

be some fraction in lowest terms. Since $0 < \frac{m}{n}$ and 0 is in F_k, let $\frac{a}{b}$ be the largest element of F_k less than $\frac{m}{n}$. Since $\frac{m}{n} < 1$ and 1 is in F_k, then there is some $\frac{p}{q}$ in F_k greater than $\frac{a}{b}$ that forms a Farey pair with it.

$$\frac{a}{b} < \frac{m}{n} < \frac{p}{q}$$

It was shown in Lemma 2, property (4), that $n \geq b + q$. By definition, $\frac{m}{n}$ is not in F_k because its denominator is greater than k. However, if $b + q < n$, then $\frac{a}{b}$ and $\frac{p}{q}$ cannot be neighbours in F_k because they would be separated by their mediant $\frac{a+p}{b+q}$. Therefore, $\frac{m}{n} = \frac{a+p}{b+q}$. By Lemma 2, property (1), $\frac{m}{n}$ forms Farey pairs with its neighbours $\frac{a}{b}$ and $\frac{p}{q}$. The only remaining case in which the induction would not yet be proven is if there were two fractions with denominator n nested between $\frac{a}{b}$ and $\frac{p}{q}$. However, as noted above, this is not possible.

Corollary 1. Any irreducible fractions $\frac{a}{b}$ and $\frac{p}{q}$, where $0 \leq \frac{a}{b} < \frac{p}{q} \leq 1$, are neighbours in $F_{\max(b,q)}$.

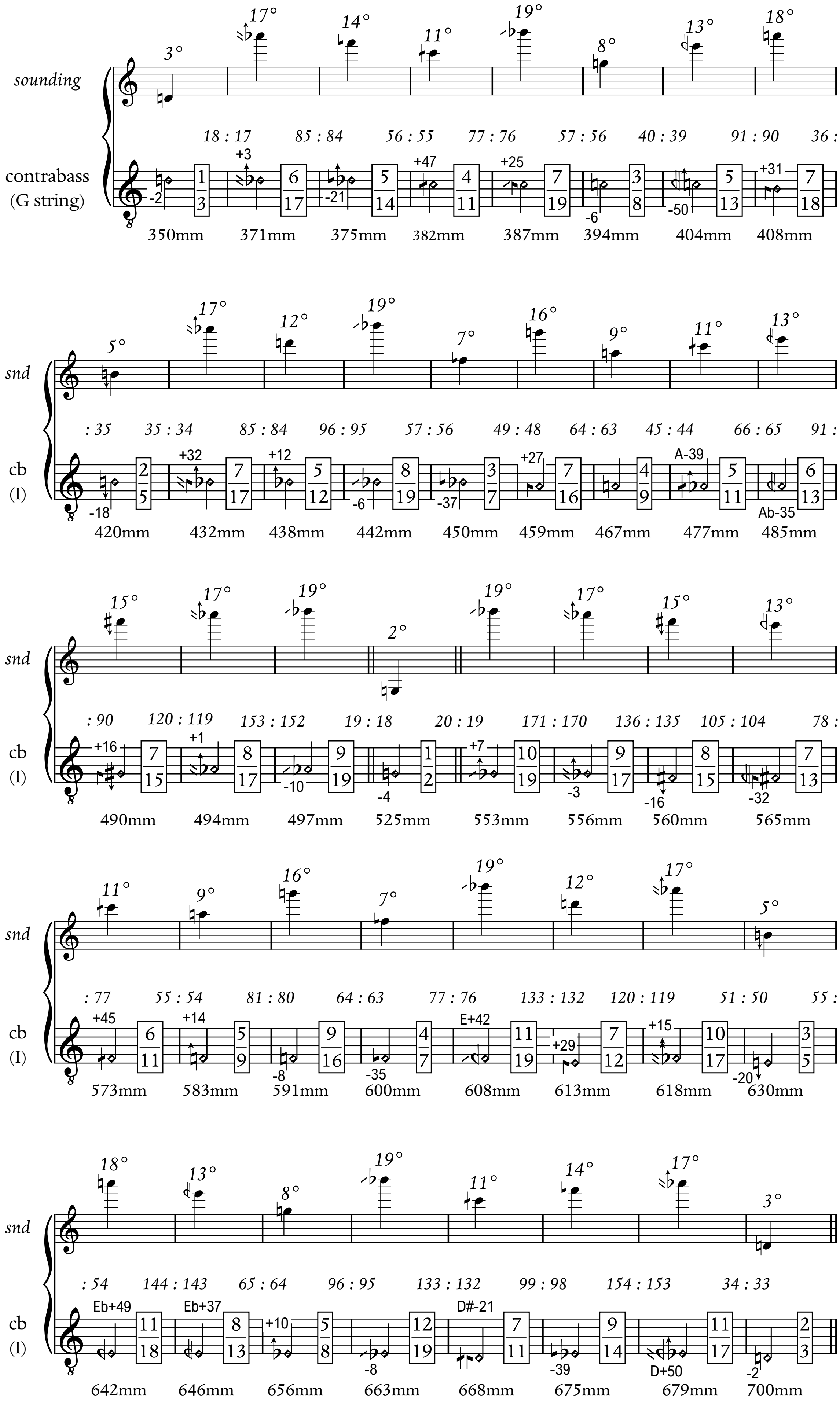
Proof. If $\frac{a}{b}$ and $\frac{p}{q}$ are not neighbours in $F_{\max(b,q)}$, then there is some irreducible fraction $\frac{m}{n}$ that lies between them where $\frac{a}{b} < \frac{m}{n} < \frac{p}{q}$ and $b + q \leq n \leq \max(b, q)$. Since $b, q > 0$, there is no such n.

Corollary 2. The Farey pair $\left(\frac{a}{b}, \frac{p}{q}\right)$ first occurring in $F_{\max(b,q)}$ will remain a Farey pair until it is separated by its mediant $\frac{a+p}{b+q}$ in F_{b+q}.

Proof. This follows from Lemma 2, properties (2) and (4).

B Playable nodes on a contrabass

Overleaf: 41 playable nodes between $\frac{1}{3}$ and $\frac{2}{3}$ (symmetric around $\frac{1}{2}$) in F_{19} notated in the *Helmholtz-Ellis JI Pitch Notation.* Distances are measured from the bridge along the G string of a contrabass with vibrating length (L) ca. 1050 mm. Melodic ratios in italics denote the microtonal intervals between the nodes if played as stopped pitches.


sounding
contrabass
(G string)
3° 17° 14° 11° 19° 8° 13° 18°
18 : 17 85 : 84 56 : 55 77 : 76 57 : 56 40 : 39 91 : 90 36 :
-2 1/3 +3 6/17 -21 5/14 +47 4/11 +25 7/19 -6 3/8 -50 5/13 +31 7/18
350mm 371mm 375mm 382mm 387mm 394mm 404mm 408mm
snd
cb
(I)
5° 17° 12° 19° 7° 16° 9° 11° 13°
: 35 35 : 34 85 : 84 96 : 95 57 : 56 49 : 48 64 : 63 45 : 44 66 : 65 91 :
-18 2/5 +32 7/17 +12 5/12 -6 8/19 -37 3/7 +27 7/16 4/9 A-39 5/11 Ab-35 6/13
420mm 432mm 438mm 442mm 450mm 459mm 467mm 477mm 485mm
snd
cb
(I)
15° 17° 19° 2° 19° 17° 15° 13°
: 90 120 : 119 153 : 152 19 : 18 20 : 19 171 : 170 136 : 135 105 : 104 78 :
+16 7/15 +1 8/17 -10 9/19 -4 1/2 +7 10/19 -3 9/17 -16 8/15 -32 7/13
490mm 494mm 497mm 525mm 553mm 556mm 560mm 565mm
snd
cb
(I)
11° 9° 16° 7° 19° 12° 17° 5°
: 77 55 : 54 81 : 80 64 : 63 77 : 76 133 : 132 120 : 119 51 : 50 55 :
+45 6/11 +14 5/9 -8 9/16 -35 4/7 E+42 11/19 +29 7/12 +15 10/17 -20 3/5
573mm 583mm 591mm 600mm 608mm 613mm 618mm 630mm
snd
cb
(I)
18° 13° 8° 19° 11° 14° 17° 3°
: 54 144 : 143 65 : 64 96 : 95 133 : 132 99 : 98 154 : 153 34 : 33
Eb+49 11/18 Eb+37 8/13 +10 5/8 -8 12/19 D#-21 7/11 -39 9/14 D+50 11/17 -2 2/3
642mm 646mm 656mm 663mm 668mm 675mm 679mm 700mm